第2回目宿題解答

3.8

(1) $P(Y=y)=P([X]=y)=P(y\leqq X<y+1)$

$$=\int_y^{y+1}\lambda e^{-\lambda x}dx=\left[-e^{-\lambda x}\right]_y^{y+1}=e^{-\lambda y}-e^{-\lambda(y+1)}$$

$$=e^{-\lambda y}(1-e^{-\lambda})$$

$$=p(1-p)^y \quad (y=0,1,\cdots)$$

これは，パラメータ $p=1-e^{-\lambda}$ の幾何分布である．

(2) $E(Y)=\frac{1-p}{p}$, $V(Y)=\frac{1-p}{p^2}$ $(p=1-e^{-\lambda})$

3.9

(1) $P(Y_d=yd)=P\left(\left[\frac{X}{d}\right]=y\right)=P\left(y\leqq\frac{X}{d}<y+1\right)$

$$=P(yd\leqq X<d(y+1))=\int_{yd}^{d(y+1)}\lambda e^{-\lambda x}dx$$

$$=\left[-e^{-\lambda x}\right]_{yd}^{d(y+1)}=e^{-\lambda yd}-e^{-\lambda d(y+1)}$$

$$=e^{-\lambda yd}(1-e^{-\lambda d})$$

$$=p(1-p)^y \quad (y=0,1,\cdots)$$

$\therefore \frac{Y_d}{d}$ は，パラメータ $p=1-e^{-\lambda d}$ の幾何分布に従う．

$E\left(\frac{Y_d}{d}\right)=\frac{1-p}{p}$ $\therefore E(Y_d)=\frac{d(1-p)}{p}$, $V\left(\frac{Y_d}{d}\right)=\frac{1-p}{p^2}$ $\therefore V(Y_d)=\frac{d^2(1-p)}{p^2}$.

$$E(Y_d) = \frac{d(1-p)}{p} = \frac{d\,e^{-\lambda d}}{1-e^{-\lambda d}} = \frac{d\left(1-\lambda d+\frac{\lambda^2 d^2}{2}+\cdots\right)}{1-\left(1-\lambda d+\frac{\lambda^2 d^2}{2}+\cdots\right)}$$

$$= \frac{d-\lambda d^2+\cdots}{\lambda d-\frac{\lambda^2 d^2}{2}+\cdots} \longrightarrow \frac{1}{\lambda} \quad (d \to 0)$$

$$V(Y_d) = \frac{d^2(1-p)}{p^2} = \frac{d^2\left(1-\lambda d+\frac{\lambda^2 d^2}{2}+\cdots\right)}{\left(\lambda d-\frac{\lambda^2 d^2}{2}+\cdots\right)^2}$$

$$= \frac{d^2-\lambda d^3+\cdots}{\lambda^2 d^2-\lambda^3 d^3+\cdots} \longrightarrow \frac{1}{\lambda^2} \quad (d \to 0)$$

3.12

(1) $G(y) = P(Y \leq y) = P(\log X \leq y) = P(X \leq e^y)$

$$= \int_{-\infty}^{e^y} f(x)\,dx = F(e^y)$$

$$\therefore \quad g(y) = G'(y) = e^y \times f(e^y) = \frac{1}{\sqrt{2\pi}\sigma} \times \frac{e^y}{e^y} \times e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$\boxed{\begin{matrix} (y = \log x) \\ (x = e^y) \end{matrix}}$

(2) $E(X^k) = E(e^{kY}) = e^{\mu k + \frac{\sigma^2}{2}k^2}$

$X = e^Y$ $Y \sim N(\mu, \sigma^2)$ のときの m.g.f.

$$\therefore \quad E(X) = e^{\mu+\frac{\sigma^2}{2}}, \quad E(X^2) = e^{2\mu+2\sigma^2}$$

$$V(X) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right).$$

(3) (2) より, $E(X^r) = e^{\mu r + \frac{\sigma^2}{2} r^2}$。

3.13

(1) $S(t) = P(T > t)$ とおくとき,

$$h(t) = \frac{f(t)}{S(t)} = -\frac{d \log S(t)}{dt}$$

$$\therefore \log S(t) = -\int_0^t h(s)ds \quad \therefore S(t) = e^{-\int_0^t h(s)ds}$$

$$P(T > s+t \mid T > s) = \frac{P(T > s+t)}{P(T > s)} = e^{-\int_0^{s+t} h(u)du} \Big/ e^{-\int_0^s h(u)du}$$

$$= e^{-\int_s^{s+t} h(u)du}$$

(2) $h(t) = bt$ のとき,

$$\int_s^{s+t} h(u)du = \frac{b}{2}\int_s^{s+t} u\,du = \frac{b}{2}\left[(s+t)^2 - s^2\right]$$

$$= \frac{b}{2}(t^2 + 2st) \geq \frac{b}{2} t^2 = \int_0^t h(u)du$$

$$\therefore \underset{P(T>s+t \mid T>s)}{\underset{\parallel}{e^{-\int_s^{s+t} h(u)du}}} \leq \underset{P(T>t)}{\underset{\parallel}{e^{-\int_0^t h(u)du}}}$$

3.14

$$S(t) = P(T>t) = 1 - F(t) = e^{-\int_0^t h(s)ds}$$

ここで

$$\int_0^t h(s)ds = \int_0^t \frac{\alpha}{\beta}\left(\frac{s}{\beta}\right)^{\alpha-1} ds = \frac{1}{\beta^\alpha}\left[s^\alpha\right]_0^t$$

$$= \frac{t^\alpha}{\beta^\alpha}$$

$$\therefore\ S(t) = e^{-t^\alpha/\beta^\alpha}$$

$$f(t) = (-S(t))' = \frac{\alpha t^{\alpha-1}}{\beta^\alpha} e^{-t^\alpha/\beta^\alpha}.$$

$$E(X^k) = \frac{\alpha}{\beta^\alpha}\int_0^\infty t^{\alpha+k-1} e^{-t^\alpha/\beta^\alpha} dt$$

$\frac{t^\alpha}{\beta^\alpha} = u$ とおくと、$t^\alpha = \beta^\alpha u$, $t = \beta u^{\frac{1}{\alpha}}$

$$dt = \frac{\beta}{\alpha} u^{\frac{1}{\alpha}-1} du$$

$$\therefore\ E(X^k) = \frac{\alpha}{\beta^\alpha}\int_0^\infty \beta^{\alpha+k-1} u^{\frac{\alpha+k-1}{\alpha}} e^{-u} \frac{\beta}{\alpha} u^{\frac{1}{\alpha}-1} du$$

$$= \beta^k \int_0^\infty u^{\frac{\alpha+k}{\alpha}-1} e^{-u} du$$

$$= \beta^k \Gamma\left(\frac{\alpha+k}{\alpha}\right)$$

$$\therefore\ E(X) = \beta\Gamma\left(\frac{\alpha+1}{\alpha}\right),\quad E(X^2) = \beta^2\Gamma\left(\frac{\alpha+2}{\alpha}\right)$$

$$V(X) = \beta^2\left[\Gamma\left(\frac{\alpha+2}{\alpha}\right) - \Gamma^2\left(\frac{\alpha+1}{\alpha}\right)\right].$$