第3回目宿題解答

5.9 $f(x)$ は a で連続だから，任意の正数 ε に対して，ある正数 δ が存在して，

$$|x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon .$$

$\Leftrightarrow$

$$|f(x) - f(a)| \geq \varepsilon \longrightarrow |x-a| \geq \delta$$

$$\therefore P(|f(X_n) - f(a)| \geq \varepsilon) \leq P(|X_n - a| \geq \delta) \to 0$$

仮定

$$\therefore P(|f(X_n) - f(a)| \geq \varepsilon) \to 0 \quad (n \to \infty)$$

任意の正数である．

5.10
$$E(\chi^2) = \sum_{i=1}^{k} \frac{E(n_i - np_i)^2}{np_i} = \sum_{i=1}^{k} \frac{np_i(1-p_i)}{np_i} = \sum_{i=1}^{k}(1-p_i)$$

$$= k-1 .$$

$$E[(\chi^2)^2] = E\left[\sum_i \sum_j \frac{(n_i - np_i)^2 (n_j - np_j)^2}{n^2 p_i p_j}\right]$$

$$= \sum_i \frac{E(n_i - np_i)^4}{n^2 p_i^2} + \sum_{i \neq j} \frac{E(n_i - np_i)^2 (n_j - np_j)^2}{n^2 p_i p_j}$$

$M_1(t) :=$ $n_i - np_i$ の MGF

$$= e^{-npt}(pe^t + 1 - p)^n \qquad (p = p_i)$$

$$\left.\frac{\partial^4 M_1}{\partial t^4}\right|_{t=0} = np(1-p)(1+3(n-2)p-3(n-2)p^2)$$

$M_2(s,t) :=$ $n_i - np_i$ と $n_j - np_j$ の同時MGF

$$= e^{-n(ps+qt)}(pe^s + qe^t + 1 - p - q)^n$$

$$(p = p_i,\ q = p_j)$$

$$\left.\frac{\partial^4 M_2}{\partial s^2 \partial t^2}\right|_{s=0,t=0} = npq\left(-1+p(2-6q)+2q+n(1-q+p(3q-1))\right)$$

$$\therefore E[(\chi^2)^2] = \sum_i \frac{np_i(1-p_i)(1+3(n-2)p_i-3(n-2)p_i^2)}{n^2p_i^2}$$

$$+ \sum_{i \neq j} \frac{np_ip_j\{-1+p_i(2-6p_j)+2p_j+n(1-p_j+p_i(3p_j-1))\}}{n^2p_ip_j}$$

$$= \sum_{i=1}^{k} \frac{1}{np_i}(1-p_i)(1+3(n-2)p_i-3(n-2)p_i^2)$$

$$+ \frac{1}{n}\left(\sum_{i,j} - \sum_{i=j}\right)\{-1+p_i(2-6p_j)+2p_j+n(1-p_j+p_i(3p_j-1))\}$$

$$= \frac{1}{n}\left[\sum\frac{1}{p_i} + 3(n-2)k - 3(n-2) - \{k+3(n-2)-3(n-2)\sum p_i^2\}\right]$$

$$+ \frac{1}{n}\Big[-k^2 + 2k - 6 + 2k + n(k^2 - k + 3 - k)$$

$$- \{-k + 2 - 6\sum p_i^2 + 2 + n(k - 1 + 3\sum p_i^2 - 1)\}\Big]$$

$$= \frac{1}{n}\Big[\sum \frac{1}{p_i} + (3n-7)k - 6(n-2) + \cancel{3(n-2)\sum p_i^2}$$

$$+ (n-1)k^2 + (-2n+4)k + 3n - 6$$

$$+ (-n+1)k + 2n - 4 - \cancel{3(n-2)\sum p_i^2} \Big]$$

$$= \frac{1}{n}\Big[\sum \frac{1}{p_i} + (n-1)k^2 - 2k - n + 2 \Big]$$

$\therefore$

$$V(X^2) = \frac{1}{n}\Big[\sum \frac{1}{p_i} + (n-1)k^2 - 2k - n + 2\Big] - (k-1)^2$$

$$= \frac{1}{n}\Big(\sum_{i=1}^{k} \frac{1}{p_i} - k^2\Big) + 2\Big(1 - \frac{1}{n}\Big)(k-1) \quad //$$

5.11

確率変数 Y_i を

$$Y_i = \begin{cases} 1 & X_i \leqq x \text{ のとき} \\ 0 & X_i > x \text{ のとき} \end{cases}$$

で定義すると、

$$Y = \#\{X_i \leqq x\} = \sum_{i=1}^{n} Y_i$$

となる。 $\therefore$ $E(Y) = n\,E(Y_i) = n\,F(x)$

$$V(Y) = n\,V(Y_i) = n\,F(x)(1-F(x))$$

$$\therefore \; E(F_n(x)) = F(x), \quad V(F_n(x)) = \frac{F(x)(1-F(x))}{n}$$

(2) $n\bar{F}_n(x) = Y \sim B_N(n, F(x))$ である．

<u>5.13</u>

$$G_n(y) = P(\max(X_1, \dots, X_n) \leqq ny)$$
$$= \prod_{i=1}^{n} P(X_i \leqq ny) = (F(ny))^n$$
$$= (1-(1-F(ny)))^n$$
$$= \left(1-\frac{n(1-F(ny))}{n}\right)^n$$

ここで，$n(1-F(ny)) \longrightarrow y^{-a} \quad (n\to\infty)$

$\therefore$ $G_n(y) \longrightarrow e^{-y^{-a}}$

(注)

$$\log G_n(y) = n\log F(ny) = n\log(1-(1-F(ny)))$$
$$= -n\left[1-F(ny) + \frac{(1-F(ny))^2}{2} + \cdots\right]$$
$$= -n(1-F(ny)) - nR_n$$

$$R_n = \frac{1}{2}(1-F(ny))^2 + \frac{1}{3}(1-F(ny))^3 + \cdots$$
$$\leqq (1-F(ny))^2 + (1-F(ny))^3 + \cdots$$
$$= \frac{(1-F(ny))^2}{F(ny)} = O\left(\frac{1}{n^2}\right)$$