7.1

P.112 を使う.

$n = 3,\quad x_k = k,\quad \mu = \frac{8}{3}$

$\sum_{k=1}^{3} k x^k = \frac{8}{3}\sum_{k=1}^{3} x^k,\quad p_k = \frac{e^{\alpha k}}{M(\alpha)} = \frac{x^k}{\sum_{k=1}^{3} x^k}$

$\Leftrightarrow\quad x + 2x^2 + 3x^3 = \frac{8}{3}(x + x^2 + x^3) \qquad \boxed{x = e^{\alpha} > 0}$

$\Leftrightarrow\quad 3x + 6x^2 + 9x^3 = 8x + 8x^2 + 8x^3$

$\Leftrightarrow\quad x^3 - 2x^2 - 5x = 0 \qquad \therefore\ x^2 - 2x - 5 = 0$

$x = 1 \pm \sqrt{6} \Rightarrow 1 + \sqrt{6}$

$M(\alpha) = \sum_{k=1}^{3} x^k = x + x^2 + x^3 = x(1 + x + x^2)$

$= (1+\sqrt{6})(1 + 1 + \sqrt{6} + 7 + 2\sqrt{6}) = (1+\sqrt{6})(9 + 3\sqrt{6})$

$= 27 + 12\sqrt{6} = 3(9 + 4\sqrt{6})$

$p_k = \frac{x^k}{M(\alpha)},\quad p_1 = \frac{(1+\sqrt{6})(4\sqrt{6}-9)}{3(4\sqrt{6}+9)(4\sqrt{6}-9)} = \frac{15 - 5\sqrt{6}}{3 \times 15} = \frac{1}{3} - \frac{\sqrt{6}}{9},$

$p_2 = \frac{(7+2\sqrt{6})(4\sqrt{6}-9)}{3(4\sqrt{6}+9)(4\sqrt{6}-9)} = \frac{-15 + 10\sqrt{6}}{3 \times 15} = \frac{2\sqrt{6}}{9} - \frac{1}{3}$

$p_3 = \frac{(19+9\sqrt{6})(4\sqrt{6}-9)}{3(4\sqrt{6}+9)(4\sqrt{6}-9)} = \frac{216 - 171 - 5\sqrt{6}}{3 \times 15} = 1 - \frac{\sqrt{6}}{9}$

エントロピーの最大値

$= \sum_{k=1}^{3} p_k \log_2 \frac{1}{p_k} = \frac{1}{3}\log_2(1+\sqrt{6}) + \log_2 \frac{9}{9-\sqrt{6}}$ //

7.2 $H(p) = -\int_0^\infty p(x)\log p(x)\,dx$

今, エントロピーが モーメント条件のみで かけるような
~~密度関数 $q(x)$ を与えると~~

$\log q(x) = -a - bx \iff q(x) = e^{-a-bx}$

$\Rightarrow \kappa e^{-\kappa x}$

$\boxed{\log q(x) = \log \kappa - \kappa x}$

∴ モーメント条件より

$\int p(x)\log\frac{1}{q(x)}\,dx = \int p(x)(a+bx)\,dx$

$= a + b\int x\,p(x)\,dx = a + b\int x\,q(x)\,dx$

$= \int q(x)(a+bx)\,dx = \int q(x)\log\frac{1}{q(x)}\,dx$

$= -\log\kappa + \kappa\int_0^\infty x\,q(x)\,dx = -\log\kappa + \frac{\kappa}{\lambda}$

$E(X) = \frac{1}{\lambda}$ より $\kappa = \lambda$

∴ エントロピー $= 1 - \log\lambda$

$0 \leqq I(P\|Q) = \int p(x)\log\frac{p(x)}{q(x)}\,dx$

$= \int p(x)\log p(x)\,dx - \int p(x)\log q(x)\,dx$

$\int p(x)\log q(x)\,dx = \int q(x)\log q(x)\,dx$

$= -H(p) + H(q)$

$\therefore H(p) \leqq H(q)$

7.3

$$H(p) = \int_a^b p(x)\log\frac{1}{p(x)}dx = E\left(\log\frac{1}{p(X)}\right)$$

$$\leq \log E\left(\frac{1}{p(X)}\right) = \log\int_a^b \frac{1}{p(x)} \times p(x)dx = \log(b-a)$$

等号は

$$p(x) = \frac{1}{b-a}$$

のとき成立. ∴ 一様分布のとき.

7.4

(1) $$-\sum_{k=0}^{\infty} p_k \log_2 p_k = -\sum_{k=0}^{\infty} p(1-p)^k\left[\log_2 p + k\log_2(1-p)\right]$$

$$= -p\log_2 p \times \frac{1}{1-(1-p)} - \log_2(1-p)\boxed{\sum_{k=0}^{\infty} k\cdot p(1-p)^k}$$

$$\boxed{} = E(X) = \frac{1-p}{p}$$

$$= -\log_2 p - \frac{1-p}{p}\log_2(1-p)$$

(2) $$-\int f(x)\log f(x)dx = -\int\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \times \left[\log\frac{1}{\sqrt{2\pi}\sigma} - \frac{(x-\mu)^2}{2\sigma^2}\right]dx$$

$$= -\log\frac{1}{\sqrt{2\pi}\sigma} + \frac{1}{2\sigma^2}V(X) = \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}$$

7.5

$$-\int\frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}\left[-\log\pi\sigma - \log\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)\right]dx$$

$$= \log\pi\sigma + \frac{1}{\pi\sigma}\int\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}\log\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)dx$$

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$$= \log \pi\sigma + \frac{1}{\pi\sigma}\int \frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}\log\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)dx$$

$\frac{x-\mu}{\sigma} = y$ とおいて

$$= \log \pi\sigma + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\log(1+y^2)\,dy$$

$$= \log \pi\sigma + \frac{2}{\pi}\int_{0}^{\infty}\frac{1}{1+y^2}\log(1+y^2)\,dy$$

$$\left(\begin{array}{ll} y = \tan x \quad (0<x<\frac{\pi}{2}), & 1+y^2 = 1+\tan^2 x = \frac{1}{\cos^2 x} \\ \tan^{-1} y = x, & dx = \frac{dy}{1+y^2} \quad \therefore dy = (1+y^2)\,dx \end{array}\right)$$

$$= \log \pi\sigma + \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\log\frac{1}{\cos^2 x}\,dx$$

$$= \log \pi\sigma - \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\log\cos x\,dx$$

ここで

$$\int_0^{\frac{\pi}{2}}\log\cos x\,dx = \int_0^{\frac{\pi}{2}}\log\sin x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\sin x\cos x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\frac{1}{2}\sin 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\sin 2x\,dx - \frac{\pi}{4}\log 2$$

$2x = t$

$$= \frac{1}{4}\int_0^{\pi}\log\sin t\,dt - \frac{\pi}{4}\log 2 = \frac{1}{2}\int_0^{\frac{\pi}{2}}\log\sin t\,dt - \frac{\pi}{4}\log 2$$

$$\therefore \int_0^{\frac{\pi}{2}}\log\cos x\,dx = -\frac{\pi}{2}\log 2$$

$$= \log\pi\sigma - \frac{4}{\pi}\times\left(-\frac{\pi}{2}\log 2\right) = \log\pi\sigma + 2\log 2.$$

7.6.
Kullback:

$$I(\theta|\tau) = \sum_{x=1}^{\infty} p_\theta(x) \log_2 \frac{p_\theta(x)}{p_\tau(x)}$$

$$\left(p_\theta(x) = \frac{1}{A(\theta)} \frac{\theta^x}{x} \text{ より} \right) \qquad A(\theta) = \log \frac{1}{1-\theta},\ A'(\theta) = \frac{1}{1-\theta}$$

$$E(X) = \frac{1}{A(\theta)} \times \frac{\theta}{1-\theta} = \frac{\theta A'(\theta)}{A(\theta)}$$

$$= \sum_{x=1}^{\infty} \frac{1}{A(\theta)} \frac{\theta^x}{x} \log_2 \left(\frac{\theta^x}{A(\theta) x} \times \frac{A(\tau) x}{\tau^x} \right)$$

$$= \frac{1}{A(\theta)} \sum_{x=1}^{\infty} \frac{\theta^x}{x} \left[x \log_2 \theta - x \log_2 \tau + \log_2 A(\tau) - \log_2 A(\theta) \right]$$

$$= \frac{1}{A(\theta)} \left[\log \theta \sum \theta^x - \log \tau \sum \theta^x + (\log A(\tau) - \log A(\theta)) \sum \frac{\theta^x}{x} \right]$$

$$= \frac{\log \theta}{A(\theta)} \times \frac{\theta}{1-\theta} - \frac{\log \tau}{A(\theta)} \frac{\theta}{1-\theta} + \log A(\tau) - \log A(\theta)$$

$$= \log \frac{\theta}{\tau} \times \frac{\theta}{A(\theta)(1-\theta)} + \log \frac{A(\tau)}{A(\theta)}$$

Fisher

$$(\log p_\theta(x))' = (x \log \theta - \log x - \log A(\theta))' =$$

$$= \frac{x}{\theta} - \frac{A'(\theta)}{A(\theta)}$$

$$E[(\log p_\theta(x))']^2 = \frac{1}{\theta^2} E\left(X - \frac{\theta A'(\theta)}{A(\theta)}\right)^2 = \frac{1}{\theta^2} V(X)$$

$$= \frac{1}{\theta^2} \times \frac{\theta (A(\theta) - \theta)}{A^2(\theta)(1-\theta)^2} = \frac{A(\theta) - \theta}{\theta A^2(\theta)(1-\theta)^2} \ //$$

7.7

(1) $\theta = \lambda + h$

$$I(\lambda\|\theta) = \sum p_\lambda(x)\log\frac{p_\lambda(x)}{p_\theta(x)} \qquad p_\lambda(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \sum e^{-\lambda}\frac{\lambda^x}{x!}\log\left(\frac{e^{-\lambda}\lambda^x}{x!}\times\frac{x!}{e^{-\theta}\theta^x}\right)$$

$$= e^{-\lambda}\sum\frac{\lambda^x}{x!}\left[-\lambda + x\log\lambda + \theta - x\log\theta\right]$$

$$= -\lambda + \log\lambda\times\lambda + \theta - \log\theta\times\lambda$$

$$= \theta - \lambda + \lambda\log\frac{\lambda}{\theta} = h - \lambda\log\left(1+\frac{h}{\lambda}\right)$$

$$\log p_\lambda(x) = -\lambda + x\log\lambda - \log x! \to (\log p_\lambda(x))' = -1 + \frac{x}{\lambda}$$

$$I(\lambda) = \frac{1}{\lambda^2}V(X) = \frac{1}{\lambda} \qquad I(\lambda\|\lambda+h) = h - \lambda\left(\frac{h}{\lambda} - \frac{h^2}{2\lambda^2} + o(h^2)\right)$$

$$= \frac{h^2}{2\lambda} + o(h^2)$$

$$\therefore \frac{1}{h^2}I(\lambda\|\lambda+h) \longrightarrow \frac{1}{2\lambda} = \frac{1}{2}I(\lambda)$$

(2)

$$I(\beta\|\gamma) = \int f_\beta(x)\log\frac{f_\beta(x)}{f_\gamma(x)}dx$$

$$\left(f_\beta(x) = \frac{1}{\Gamma(\alpha)\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\frac{x}{\beta}}\right)$$

$$= \int\frac{1}{\Gamma(\alpha)\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\frac{x}{\beta}}\log\left(\frac{\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\frac{x}{\beta}}\,\Gamma(\alpha)\gamma}{\Gamma(\alpha)\beta\left(\frac{x}{\gamma}\right)^{\alpha-1}e^{-\frac{x}{\gamma}}}\right)dx$$

$$= \int\frac{1}{\Gamma(\alpha)\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\frac{x}{\beta}}\left[-(\alpha-1)\log\beta - \frac{x}{\beta} + \log\gamma + (\alpha-1)\log\gamma + \frac{x}{\gamma} - \log\beta\right]dx$$

$$= -\alpha\log\beta + \alpha\log\gamma - \left(\frac{1}{\beta}-\frac{1}{\gamma}\right)E(X) = \alpha\log\frac{\gamma}{\beta} - \left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\times\alpha\beta$$

$$\log f_\beta(x) = (\alpha-1)\log x - \alpha\log\beta - \frac{x}{\beta} - \log\Gamma(\alpha)$$

$$(\log f_\beta(x))' = -\frac{\alpha}{\beta} + \frac{x}{\beta^2} = \frac{1}{\beta^2}(x - \alpha\beta)$$

$$I(\beta) = \frac{1}{\beta^4}V(X) = \frac{1}{\beta^4}\times\alpha\beta^2 = \frac{\alpha}{\beta^2}$$

$$I(\beta\|\beta+h) = \alpha\log\left(1+\frac{h}{\beta}\right) - \frac{\alpha h}{\beta}\left(1+\frac{h}{\beta}\right)^{-1} = \frac{\alpha h^2}{2\beta^2} + o(h^2)$$

$$\therefore \frac{1}{h^2}I(\beta\|\beta+h) \longrightarrow \frac{\alpha}{2\beta^2} = \frac{1}{2}I(\beta)$$

7.8

$$\frac{1}{h^2}\int\left(\sqrt{f_{\theta+h}(x)}-\sqrt{f_\theta(x)}\right)^2 dx \text{ において}$$

$$\frac{\sqrt{f_{\theta+h}(x)}-\sqrt{f_\theta(x)}}{h} = \frac{f_{\theta+h}(x)-f_\theta(x)}{h\left(\sqrt{f_{\theta+h}(x)}+\sqrt{f_\theta(x)}\right)} \longrightarrow \frac{f'_\theta(x)}{2\sqrt{f_\theta(x)}} \quad (h\to 0)$$

$$\therefore \frac{1}{h^2}\int\left(\sqrt{f_{\theta+h}(x)}-\sqrt{f_\theta(x)}\right)^2 dx \longrightarrow \int \frac{\left(f'_\theta(x)\right)^2}{4f_\theta(x)}dx$$

$$=\frac{1}{4}\int\left(\frac{f'_\theta(x)}{f_\theta(x)}\right)^2\times f_\theta(x)\,dx = \frac{1}{4}E\left[\left(\log f_\theta'(x)\right)^2\right] = \frac{1}{4}I(\theta)$$

7.9

(1)

$$1=\int_0^\infty f_\theta(x)\,dx = c\times\int_0^\infty e^{-\frac{\theta^2x^2}{\pi}}dx$$

$$\left(\frac{\theta^2x^2}{\pi}=t \text{ とおくと} \qquad \frac{2\theta^2x}{\pi}dx=dt \qquad x=\frac{\sqrt{t\pi}}{\theta}\right.$$

$$= c\times\int_0^\infty e^{-t}\times\frac{\pi}{2\theta^2}\times\frac{\theta}{\sqrt{t\pi}}dt$$

$$=\frac{c\sqrt{\pi}}{2\theta}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt = \frac{c\sqrt{\pi}}{2\theta}\times\sqrt{\pi} = \frac{c}{2\theta}\pi$$

$$\therefore c=\frac{2\theta}{\pi} \ /\!/$$

(2)

$$E(X)=\int_0^\infty\frac{2\theta}{\pi}\times x\, e^{-\frac{\theta^2x^2}{\pi}}dx = \frac{2\theta}{\pi}\int_0^\infty e^{-t}\times\frac{\pi}{2\theta^2}dt = \frac{1}{\theta}\int_0^\infty e^{-t}dt = \frac{1}{\theta}$$

$$E(X^2)=\int_0^\infty\frac{2\theta}{\pi}x^2e^{-\frac{\theta^2x^2}{\pi}}dx = \frac{2\theta}{\pi}\int_0^\infty\frac{\sqrt{t\pi}}{\theta}\times\frac{\pi}{2\theta^2}e^{-t}dt$$

$$=\frac{\sqrt{\pi}}{\theta^2}\int_0^\infty t^{\frac{3}{2}-1}e^{-t}dt = \frac{\sqrt{\pi}}{\theta^2}\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{\theta^2}\times\frac{1}{2}\sqrt{\pi} = \frac{\pi}{2\theta^2}$$

$$\therefore V(X)=\frac{\pi}{2\theta^2}-\frac{1}{\theta^2}=\left(\frac{\pi}{2}-1\right)\frac{1}{\theta^2}$$

(3)

$$\log f_\theta(x) = \log 2\theta - \log \pi - \frac{\theta^2 x^2}{\pi} \qquad E(x^2) = \frac{\pi}{2\theta^2}$$

$$\left(\log f_\theta(x)\right)' = \frac{1}{\theta} - \frac{2\theta x^2}{\pi}$$

$$E(X^4) = \int_0^\infty \frac{2\theta}{\pi} x^4 e^{-\frac{\theta^2 x^2}{\pi}} dx \qquad \frac{\theta^2 x^2}{\pi} = t$$

$$\frac{2\theta^2 x}{\pi} dx = dt$$

$$= \frac{2\theta}{\pi} \int_0^\infty \frac{\pi^2}{\theta^4} t^2 \times e^{-t} \times \frac{\pi}{2\theta^2} \times \frac{\theta}{\sqrt{t\pi}} dt$$

$$= \frac{\pi\sqrt{\pi}}{\theta^4} \int_0^\infty t^{\frac{5}{2}-1} e^{-t} dt = \frac{\pi\sqrt{\pi}}{\theta^4} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{\pi\sqrt{\pi}}{\theta^4} \times \frac{3\sqrt{\pi}}{4} = \frac{3\pi^2}{4\theta^4}$$

$$E\left[\left(\log f_\theta(x)\right)'^2\right] = \frac{1}{\theta^2} - \frac{4}{\pi} \times \frac{\pi}{2\theta^2} + \frac{4\theta^2}{\pi^2} \times \frac{3\pi^2}{4\theta^4}$$

$$= \frac{2}{\theta^2} = I(\theta)$$

$$\left(\log f_\theta(x)\right)'' = -\frac{1}{\theta^2} - \frac{2x^2}{\pi}$$

$$-E\left[\left(\log f_\theta(x)\right)''\right] = \frac{1}{\theta^2} + \frac{2}{\pi} \times \frac{\pi}{2\theta^2} = \frac{2}{\theta^2} = I(\theta)$$